THE RECURSION OPERATOR FOR A CONSTRAINED CKP HIERARCHY

CHUANZHONG LI†, KELEI TIAN †, JINGSONG HE*†‡, YI CHENG †

†*Department of Mathematics, USTC, Hefei, 230026 Anhui, P. R. China*
‡*Department of Mathematics, NBU, Ningbo, 315211 Zhejiang, P. R. China*

Abstract. This paper gives a recursion operator for a 1-constrained CKP hierarchy, and by the recursion operator it proves that the 1-constrained CKP hierarchy can be reduced to the mKdV hierarchy under condition $q = r$.

MR(2000) Subject Classification: 37K05, 37K10, 35Q53.
Keywords: recursion operator, constrained CKP hierarchy, mKdV hierarchy.

1. Introduction

It is well known that conserved quantities are closely related to symmetries of equations, and possessing infinite number of conserved quantities or symmetries is a common property of the classical integrable systems. There are many results on finding concrete forms of them [1, 2, 3]. Recursion operator is one kind of effective tools to generate symmetries of the classical integrable systems[4, 5]. On the other hand, recursion operator is also used to establish the Hamiltonian structure of the classical systems [1, 6, 7]) and integrable flows of curves [8]. So it is vital to construct the recursion operator for the classical systems. In the papers [7, 9, 10, 11], several different methods are used to construct recursion operators. Furthermore, it is highly non-trivial to reduce some results from constrained KP (cKP) to constrained BKP (cBKP) and constrained CKP (cCKP) hierarchies which can be seen from bilinear forms [12, 13, 14] and gauge transformations [15]. In the paper [10], the recursion operator for a 1-constrained cBKP hierarchy has been given. So the purpose of this paper is to give the recursion operator of a cCKP hierarchy and to show the relation between cCKP hierarchy and mKdV hierarchy.

The organization of this paper is as follows. We recall some basic facts for the KP hierarchy and a constrained CKP hierarchy in section 2. In section 3, the recursion operator for the cCKP hierarchy is discussed and used to generate the

* Corresponding author: jshe@ustc.edu.cn, hejingsong@nbu.edu.cn
Supported by: NSF of China under Grant No. 10671187 and 10971109, and the Program for NCET under Grant No. NECT-08-0515.

t_3 flows and t_5 flows, which are consistent with results given by eigenfunction equations of this sub-hierarchy. Meanwhile we will show that the t_3 flows and t_5 flows are the 2-component generalization of mKdV equation and 5th order mKdV equation. Section 4 is devoted to conclusions and discussions in which we will describe the reducing relations of cKP hierarchy, cBKP hierarchy, cCKP hierarchy, KdV hierarchy and mKdV hierarchy.

2. The constrained CKP hierarchy

Since its introduction in 1980s, the KP hierarchy [16, 17] is one of the most important research topics in the area of classical integrable systems. The KP hierarchy is constructed by the pseudo-differential operator $L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \ldots.$ like this:

$$L_{t_n} = [B_n, L],$$

where $B_n = (L^n)_+$. The t_2 (denoted by y) flows and t_3(denoted by t) flows imply the KP equation

$$(4u_t - u_{xxx} - 12uu_x)_x - 3u_{yy} = 0 \tag{2.1}$$

where $u = u_2$. The eigenfunction q and conjugate eigenfunction r of KP hierarchy are defined by

$$q_{t_m} = B_m q, \quad r_{t_m} = -B_m^* r. \tag{2.2}$$

It is well known that there are two kinds of sub-hierarchies of KP hierarchy, i.e. BKP hierarchy [16] and CKP hierarchy [18]. In order to define the CKP hierarchy, we need a formal adjoint operation $*$ for an arbitrary pseudo-differential operator $P = \sum\limits_i p_i\partial^i$, $P^* = \sum\limits_i (-1)^i\partial^i p_i$. For example, $\partial^* = -\partial$, $(\partial^{-1})^* = -\partial^{-1}$, and $(AB)^* = B^*A^*$ for two operators. The CKP hierarchy is a reduction of the KP hierarchy by the constraint

$$L^* = -L, \tag{2.3}$$

which compresses all even flows of the KP hierarchy, i.e. the Lax equation of the CKP hierarchy has only odd flows,

$$\frac{\partial L}{\partial t_{2n+1}} = [B_{2n+1}, L], \ n = 0, 1, 2, ..., \tag{2.4}$$

which indicates $u_i = u_i(t_1, t_3, t_5, ...)$ for the CKP hierarchy. This hierarchy contains the $(2+1)$ dimensional CKP equation:

$$9v_{x,t_5} - 5v_{t_3t_3} + (v_{xxxxx} + 15v_xv_{xxx} + 15v_x^3 - 5v_{xx,t_3} - 15v_xv_{t_3} + \frac{45}{4}v_{xx}^2)_x = 0, \tag{2.5}$$

where $v = \int u_2$. Let $v_{t_3} = 0$, eq.(2.5) becomes a well-known equation called Kaup-Kupershmidt equation [19, 20]

$$9u_{t_5} + (u_{xxxx} + 15uu_{xx} + 15u^3 + \frac{45}{4}u_x^2)_x = 0, \tag{2.6}$$

where $u = u_2$.
Moreover, the so called "constrained KP hierarchy" (cKP) [9, 21, 22]) is a very interesting sub-hierarchy developed from the point of view of symmetry constraint, and the Lax operator for 1-constrained KP is given by

$$L = \partial + \sum_{i=1}^{n} q_i \partial^{-1} r_i, \tag{2.7}$$

here q_i (r_i) is the eigenfunction(conjugation eigenfunction) of L in eq.(2.7). By considering CKP condition on the constrained KP hierarchy, i.e. $L^* = -L$, the constrained CKP hierarchy (cCKP) can be defined through a following Lax operator [23]

$$L = \partial + \sum_{i=1}^{n} (q_i \partial^{-1} r_i + r_i \partial^{-1} q_i). \tag{2.8}$$

In the following context, we take $n = 1$ for simplicity, i.e.,

$$\mathcal{L} = \partial + q\partial^{-1} r + r\partial^{-1} q. \tag{2.9}$$

Note that q and r satisfy the eigenfunction eqs.(2.2) associated with $\mathcal{L}$ in eq.(2.9).

As we know, the evolutions of CKP hierarchy with respect to $t_2, t_4, t_6, ...$ are freezed. They are also done to **1-constrained CKP hierarchy** whose evolution equations are like this:

$$\frac{\partial \mathcal{L}}{\partial t_{2n+1}} = [B_{2n+1}, \mathcal{L}], \ n = 0, 1, 2, ... \tag{2.10}$$

In order to get the explicit form of the flow equations, we need B_{2n+1},

$$\begin{aligned}
B_1 &= \partial, \\
B_3 &= \partial^3 + 6qr\partial + 3rq_x + 3qr_x, \\
B_5 &= \partial^5 + 10qr\partial^3 + (15rq_x + 15qr_x)\partial^2 + (15qr_{xx} + 15rq_{xx} + 40q^2r^2 + 20q_xr_x)\partial \\
&\quad +40qr^2q_x + 40rq^2r_x + 5qr_{xxx} + 5rq_{xxx} + 10q_xr_{xx} + 10r_xq_{xx}.
\end{aligned}$$

After a direct computation from eigenfunction eqs.(2.2), we can get the first few flows of the cCKP hierarchy

$$\begin{cases} q_{t_1} = q_x \\ r_{t_1} = r_x, \end{cases} \tag{2.11}$$

$$\begin{cases} q_{t_3} = q_{xxx} + 9qrq_x + 3q^2 r_x \\ r_{t_3} = r_{xxx} + 9qrr_x + 3r^2 q_x, \end{cases} \tag{2.12}$$

$$\begin{cases} q_{t_5} & = q_{xxxxx} + 15qrq_{xxx} + 30rq_x q_{xx} + 25qr_x q_{xx} + 25qq_x r_{xx} \\ & +80q^2r^2q_x + 20q_x r_x q_x + 40rq^3 r_x + 5q^2 r_{xxx} \\ r_{t_5} & = r_{xxxxx} + 15qrr_{xxx} + 30qr_x r_{xx} + 25rq_x r_{xx} + 25rr_x q_{xx} \\ & +80q^2r^2r_x + 20q_x r_x r_x + 40qr^3 q_x + 5r^2 q_{xxx}. \end{cases} \tag{2.13}$$

Let $q = r$, eq.(2.12) implies mKdV equation

$$q_{t_3} = q_{xxx} + 12q^2 q_x. \tag{2.14}$$

A transformation $q = \frac{\sqrt{3}u}{6}$ leads it to the form of mKdV equation in ([24])

$$u_{t_3} = u_{xxx} + u^2 u_x. \tag{2.15}$$

Let $q = r$, eq.(2.13) implies 5th order mKdV equation

$$q_{t_5} = q_{xxxxx} + 20q^2 q_{xxx} + 80qq_x q_{xx} + 120q^4 q_x + 20(q_x)^3. \tag{2.16}$$

A transformation $q = \frac{\sqrt{3}u}{6}, t_5 = t$ leads it to the standard 5th order mKdV equation in [24]

$$u_t = u_{xxxxx} + \frac{5}{3}u^2 u_{xxx} + \frac{20}{3}uu_x u_{xx} + \frac{5}{6}u^4 u_x + \frac{5}{3}(u_x)^3. \tag{2.17}$$

Note that there exist other forms of mKdV equation and 5th order mKdV equation, for example [23, 25]. It is very difficult to observe recursion operator from equations on t_3 flows and t_5 flows above. We shall find it in next section from eigenfunction equations on q and r, and may use it to generate any higher order flows. To illustrate the validity of recursion operator, we shall use it to generate t_3 flows from trivial flows, i.e. t_1 flows, and further generate t_5 flows from t_3 flows.

3. Recursion operator

In this section, we will give the form of recursion operator R. Now, we define the following four operators:

$$\begin{aligned} R_{11} &= \mathcal{L}^2 + 3qr + \mathcal{L}(r)\partial^{-1}q + 2q_x\partial^{-1}r - q\partial^{-1}qr\partial^{-1}r - r\partial^{-1}q\partial - r\partial^{-1}q^2\partial^{-1}r \\ &\quad -2r\partial^{-1}rq\partial^{-1}q - r\partial^{-1}q(\int rq) - q\partial^{-1}q(\int r^2), \\ R_{12} &= 2q_x\partial^{-1}q + 3q^2 - 2q\partial^{-1}q^2\partial^{-1}r - q\partial^{-1}q\partial - q\partial^{-1}qr\partial^{-1}q - r\partial^{-1}q^2\partial^{-1}q \\ &\quad -q\partial^{-1}q(\int rq) - r\partial^{-1}q(\int q^2) + \mathcal{L}(q)\partial^{-1}q, \\ R_{21} &= 2r_x\partial^{-1}r + 3r^2 - 2r\partial^{-1}r^2\partial^{-1}q - r\partial^{-1}r\partial - r\partial^{-1}qr\partial^{-1}r - q\partial^{-1}r^2\partial^{-1}r \end{aligned}$$

$$
\begin{aligned}
& -r\partial^{-1}r(\int rq) - q\partial^{-1}r(\int r^2) + \mathcal{L}(r)\partial^{-1}r, \\
R_{22} &= \mathcal{L}^2 + 3qr + \mathcal{L}(q)\partial^{-1}r + 2r_x\partial^{-1}q - r\partial^{-1}qr\partial^{-1}q - q\partial^{-1}r\partial - q\partial^{-1}r^2\partial^{-1}q \\
& -2q\partial^{-1}qr\partial^{-1}r - q\partial^{-1}r(\int rq) - r\partial^{-1}r(\int q^2).
\end{aligned}
$$

Theorem 3.1. *The recursion relation of flows for the 1-cCKP hierarchy (2.10) is like this:*

$$
\begin{pmatrix} q \\ r \end{pmatrix}_{\mathrm{t_{m+2}}} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix}_{\mathrm{t_m}}. \tag{3.1}
$$

Proof. Using the identities ([26]) below

$$
\begin{aligned}
(B_n f\partial^{-1}g)_- &= B_n(f)\partial^{-1}g, && (3.2)\\
(f\partial^{-1}gB_n)_- &= f\partial^{-1}B_n^*(g), && (3.3)\\
f_1\partial^{-1}g_1 \cdot f_2\partial^{-1}g_2 &= f_1(\int g_1f_2)\partial^{-1}g_2 - f_1\partial^{-1}g_2(\int f_2g_1), && (3.4)
\end{aligned}
$$

we can calculate the $\mathcal{L}^2$ as following:

$$
\mathcal{L}^2 = \partial^2 + 4qr + q\partial^{-1}\mathcal{L}^*(r) + r\partial^{-1}\mathcal{L}^*(q) + \mathcal{L}(q)\partial^{-1}r + \mathcal{L}(r)\partial^{-1}q \tag{3.5}
$$

with

$$
\mathcal{L}(q) = q_x + q\int rq + r\int q^2, \quad \mathcal{L}(r) = r_x + q\int r^2 + r\int qr,
$$

$$
\mathcal{L}^*(q) = -q_x - q\int rq - r\int q^2, \quad \mathcal{L}^*(r) = -r_x - q\int r^2 - r\int qr.
$$

Therefore,

$$
B_2 = \partial^2 + 4qr.
$$

Denote A_n as $(\mathcal{L}^n)_-$, $n = 1, 2, \ldots$. Considering CKP condition and eqs.(2.2), q and r should satisfy the same equation, i.e.

$$
B_m(q) = q_{t_m}, \quad B_m(r) = r_{t_m}, \tag{3.6}
$$

then

$$
q_{t_{m+2}} = (\mathcal{L}^2\mathcal{L}^m)_+q = B_2B_mq + (B_2A_m)_+q + (A_2B_m)_+q, \tag{3.7}
$$

$$
r_{t_{m+2}} = (\mathcal{L}^2\mathcal{L}^m)_+r = B_2B_mr + (B_2A_m)_+r + (A_2B_m)_+r. \tag{3.8}
$$

Firstly, we will calculate $(B_2A_m)_+$. Now, we set $A_m = \partial^{-1}a_1 + \partial^{-2}a_2 + \ldots$. So, $(B_2A_m)_+ = \partial a_1 + a_2$. The identity $Res_\partial[\mathcal{L}^m, \mathcal{L}] = 0$ yields:

$$
Res_\partial[B_m, \mathcal{L}] = Res_\partial[-A_m, \mathcal{L}] = Res_\partial[-A_m, B_1]. \tag{3.9}
$$

The first residue of eq.(3.9) equals $Res_\partial \mathcal{L}_{t_m} = 2(qr)_{t_m}$, the last residue of eq.(3.9) yields $Res_\partial[\partial, \partial^{-1}a_1 + \partial^{-2}a_2 + ...] = a_{1x}$. So,

$$a_1 = \int 2(qr)_{t_m}. \tag{3.10}$$

To compute a_2, we should use identity $Res_\partial[\mathcal{L}^m, \mathcal{L}^2] = 0$, considering the similar identity

$$Res_\partial[B_m, \mathcal{L}^2] = Res_\partial[-A_m, \mathcal{L}^2] = Res_\partial[-A_m, B_2]. \tag{3.11}$$

The first residue of eq.(3.11) equals $Res_\partial \mathcal{L}^2_{t_m} = 0$, the last residue of eq.(3.11) yields $Res_\partial[\partial^2 + 4qr, \partial^{-1}a_1 + \partial^{-2}a_2 + ...] = -a_{1xx} + 2a_{2x}$. We can easily get

$$a_2 = a_{1x}/2 = (qr)_{t_m}. \tag{3.12}$$

Hence,

$$(B_2A_m)_+ = \partial \cdot \int 2(qr)_{t_m} + (qr)_{t_m}. \tag{3.13}$$

About the term $(A_2B_m)_+$, we write it as $A_2B_m - (A_2B_m)_-$. The first term is relevant to t_m flow. Using the identity(3.3), we can compute the second term

$$\begin{aligned}
(A_2B_m)_- &= [(q\partial^{-1}\mathcal{L}^*(r) + r\partial^{-1}\mathcal{L}^*(q) + \mathcal{L}(q)\partial^{-1}r + \mathcal{L}(r)\partial^{-1}q)B_m]_- \\
&= q\partial^{-1}B_m^*\mathcal{L}^*(r) + r\partial^{-1}B_m^*\mathcal{L}^*(q) + \mathcal{L}(q)\partial^{-1}B_m^*(r) + \mathcal{L}(r)\partial^{-1}B_m^*(q).
\end{aligned}$$

Considering eqs.(3.6),

$$\begin{aligned}
B_m^*\mathcal{L}^*(q) &= \mathcal{L}^*B_m^*(q) + [B_m^*, \mathcal{L}^*](q) \\
&= \mathcal{L}B_m(q) + [B_m, \mathcal{L}](q) \\
&= \mathcal{L}(q_{t_m}) + \mathcal{L}_{t_m}(q) \\
&= q_{xt_m} + r\int qq_{t_m} + q\int rq_{t_m} \\
&\quad +(r_{t_m}\partial^{-1}q + r\partial^{-1}q_{t_m} + q_{t_m}\partial^{-1}r + q\partial^{-1}r_{t_m})(q) \\
&= q_{xt_m} + 2r\int qq_{t_m} + q\int rq_{t_m} + r_{t_m}\int q^2 + q_{t_m}\int rq + q\int r_{t_m}q.
\end{aligned}$$

Similarly, we can get

$$B_m^*\mathcal{L}^*(r) = r_{xt_m} + 2q\int rr_{t_m} + r\int qr_{t_m} + q_{t_m}\int r^2 + r_{t_m}\int qr + r\int q_{t_m}r.$$

After bringing these results into eq.(3.7), we get the recursion flow of q

$$\begin{aligned}
q_{t_{m+2}} = &\Big[\mathcal{L}^2 + 3qr + \mathcal{L}(r)\partial^{-1}q + 2q_x\partial^{-1}r - q\partial^{-1}qr\partial^{-1}r - r\partial^{-1}q\partial - r\partial^{-1}q^2\partial^{-1}r \\
&-2r\partial^{-1}rq\partial^{-1}q - r\partial^{-1}q(\int rq) - q\partial^{-1}q(\int r^2)\Big]q_{t_m} \\
&+\Big[2q_x\partial^{-1}q + 3q^2 - 2q\partial^{-1}q^2\partial^{-1}r - q\partial^{-1}q\partial - q\partial^{-1}qr\partial^{-1}q - r\partial^{-1}q^2\partial^{-1}q
\end{aligned}$$

$$-q\partial^{-1}q(\int rq) - r\partial^{-1}q(\int q^2) + \mathcal{L}(q)\partial^{-1}q\Big]r_{t_m}.$$

Similarly after bringing these results into eq.(3.8), we get the recursion flow of r

$$\begin{aligned}
r_{t_{m+2}} &= \Big[2r_x\partial^{-1}r + 3r^2 - 2r\partial^{-1}r^2\partial^{-1}q - r\partial^{-1}r\partial - r\partial^{-1}qr\partial^{-1}r - q\partial^{-1}r^2\partial^{-1}r \\
&\quad -r\partial^{-1}r(\int rq) - q\partial^{-1}r(\int r^2) + \mathcal{L}(r)\partial^{-1}r\Big]q_{t_m} \\
&\quad +\Big[\mathcal{L}^2 + 3qr + \mathcal{L}(q)\partial^{-1}r + 2r_x\partial^{-1}q - r\partial^{-1}qr\partial^{-1}q - q\partial^{-1}r\partial - q\partial^{-1}r^2\partial^{-1}q \\
&\quad -2q\partial^{-1}qr\partial^{-1}r - q\partial^{-1}r(\int rq) - r\partial^{-1}r(\int q^2)\Big]r_{t_m}.
\end{aligned}$$

Then we get the recursion operator written in eq.(3.1). □

Now, let us inspect whether the results from this recursion operator are consistent with what from the eigenfunction eqs.(2.2).

By a very tedious calculation, we have checked that they are consistent on the t_3 flows and t_5 flows. Of course we can generate the t_7 flows, t_9 flows etc. in the same way which should be also consistent with the corresponding flows from Sato's methods.

Corollary 3.2. *The 1-constrained CKP hierarchy (2.10) can be reduced to the mKdV hierarchy by condition $q = r$.*

Proof. Let $q = r$, we can get

$$\begin{aligned}
q_{t_{m+2}} &= \Big[\mathcal{L}^2(q,q) + 3qq + \mathcal{L}(q)\partial^{-1}q + 2q_x\partial^{-1}q - q\partial^{-1}qq\partial^{-1}q - q\partial^{-1}q\partial - q\partial^{-1}q^2\partial^{-1}q \\
&\quad -2q\partial^{-1}qq\partial^{-1}q - q\partial^{-1}q(\int qq) - q\partial^{-1}q(\int q^2)\Big]q_{t_m} \\
&\quad +\Big[2q_x\partial^{-1}q + 3q^2 - 2q\partial^{-1}q^2\partial^{-1}q - q\partial^{-1}q\partial - q\partial^{-1}qq\partial^{-1}q - q\partial^{-1}q^2\partial^{-1}q \\
&\quad -q\partial^{-1}q(\int qq) - q\partial^{-1}q(\int q^2) + \mathcal{L}(q)\partial^{-1}q\Big]q_{t_m} \\
&= \Big[\partial^2 + 4qq + q\partial^{-1}(-q_x - q\int q^2 - q\int qq) + q\partial^{-1}(-q_x - q\int qq - q\int q^2) \\
&\quad +2(q_x + q\int qq + q\int q^2)\partial^{-1}q + (q_x + q\int q^2 + q\int qq)\partial^{-1}q + 3qq \\
&\quad +2q_x\partial^{-1}q - q\partial^{-1}qq\partial^{-1}q - q\partial^{-1}q\partial - q\partial^{-1}q^2\partial^{-1}q \\
&\quad -2q\partial^{-1}qq\partial^{-1}q - q\partial^{-1}q(\int qq) - q\partial^{-1}q(\int q^2)\Big]q_{t_m}
\end{aligned}$$

$$
\begin{aligned}
&+\Big[2q_x\partial^{-1}q+3q^2-2q\partial^{-1}q^2\partial^{-1}q-q\partial^{-1}q\partial-q\partial^{-1}qq\partial^{-1}q-q\partial^{-1}q^2\partial^{-1}q\\
&-q\partial^{-1}q(\int qq)-q\partial^{-1}q(\int q^2)+(q_x+q\int qq+q\int q^2)\partial^{-1}q\Big]q_{t_m}\\
=\;&\Big[\partial^2+10q^2+q\partial^{-1}(-2q_x-8q\int q^2-8qq\partial^{-1}q-2q\partial)+(8q_x\\
&+8q\int qq)\partial^{-1}q\Big]q_{t_m}\\
=\;&(\partial^2+8q^2+8q_x\partial^{-1}q)q_{t_m}.
\end{aligned}
$$

Then we can get the reduced recursion operator which is just the recursion operator for mKdV hierarchy

$$R_r \;=\; \partial^2+8q^2+8q_x\partial^{-1}q. \tag{3.14}$$

The same transformation $q=\frac{\sqrt{3}u}{6}$ leads to the form of mKdV hierarchy in [24]

$$\mathbf{R} \;=\; \partial^2+\frac{2}{3}u^2+\frac{2}{3}u_x\partial^{-1}u. \tag{3.15}$$

So we can get the whole mKdV hierarchy from the trivial flow under the condition $q=r$. For example, mKdV eq.(2.15) and 5th order mKdV eq.2.16 can be got from this cCKP hierarchy. Now we will say that condition $q=r$ can reduce the cCKP hierarchy to mKdV hierarchy. □

4. Conclusions and Discussions

The recursion operator in eq.(3.1) for a cCKP system was found from the eigenfunction equations on q and r. This operator was used to generate t_3 flows (eqs.(2.12)) and t_5 flows (eqs.(2.13)) from the t_1 flows of this special hierarchy, which are consistent with flows got from eigenfunction eqs.(3.6). That demonstrated the validity of the recursion operator. Of course one can also use it to generate higher order flows. On the other hand, our results are more complicated than recursion operator for cKP hierarchy [9]. Moreover, we can also get the following reduction chain from corollary 3.2:

$$\text{cKP hierarchy} \xrightarrow{L^*=-L} \text{cCKP hierarchy} \xrightarrow{q=r} \text{mKdV hierarchy}. \tag{4.1}$$

Similarly, the KdV hierarchy will appear in the reduction of cBKP hierarchy [10]. As we know, the relationship of KdV hierarchy and mKdV hierarchy can be represented by miura transformation, but what is the similar transformation between cBKP hierarchy and cCKP hierarchy. In [27], the relationship of KdV hierarchy and mKdV hierarchy can be seen from the decomposition of differential Lax operator, but whether the relationship of cBKP hierarchy and

cCKP hierarchy can be comprehended from the the decomposition of pseudo-differential Lax operator is still unknown and interesting.

Acknowledgments:
We thank Professor Li Yishen (USTC, China) for long-term encouragements and supports.

References

[1] Fokas A S, Santini P M. The recursion operator of the Kadomtsev-Petviashvili equation and the squared eigenfunctions of the Schrödinger operator. Stud Appl Math, 1986, 75: 179-185

[2] Fokas A S, Santini P M. Recursion operators and bi-Hamiltonian structures in multidimensions II. Commun. Math. Phys. 116(1988), 449-474

[3] Matsukidaira J, Satsuma J, Strampp W. Conserved quantities and symmetries of KP hierarchy. J Math Phys, 1990, 31: 1426-1434

[4] Fuchssteiner B, Fokas A S. Symplectic structures, their Bäcklund transformation and hereditary symmetries. Phys D, 1981, 4: 47-66

[5] Olver P J. Applications of Lie groups to differential equations. New York: Springer-Verlag, 1986

[6] Santini P M, Fokas A S. Recursion operators and bi-Hamiltonian structures in multidimensions I. Commun Math Phys, 1988, 115: 375-419

[7] Strampp W, Oevel W. Recursion operators and Hamitonian structures in Sato theory. Lett Math Phys, 1990, 20: 195-210

[8] Stephen C A. Bi-Hamiltonian operators, integrable flows of curves using moving frames and geometric map equations. J Phys A: Math Gen, 2006, 39: 2043-2072

[9] Cheng Y. Constraints of the Kadomtsev-Petviashvili hierarchy. J Math Phys, 1992, 33: 3774-3782

[10] Loris I. Recursion operator for a constraint BKP system, In: Boiti M, Martina L, etal ed. Proceedings of the Workshop on Nonlinearity, Integrability and All That Twenty years After NEEDS'79. Singapore: World Scientific, 1999. 325-330

[11] Gurses M, Karasu A, Sokolov V V. On construction of recursion operators from Lax representation. J Math Phys, 1999, 40: 6473-6490

[12] Loris I, Willox R. Bilinear form and solutions of the k-constrained Kadomtsev-Petviashvili hierarchy. Inverse Problems, 1997, 13: 411-420

[13] Loris I. Symmetry reductions of the BKP hierarchy. J Math Phys, 1999, 40: 1420-1431

[14] Loris I. Bilinear representation of integrable equation. Theor Math Phys, 2002, 133: 1549-1556

[15] He J S, Wu Z W, Cheng Y. Gauge transformations for the constrained CKP and BKP hierarchies. J Math Phys, 2007, 48: 113-519

[16] Date E, Kashiwara M, Jimbo M, Miwa T. Transformation groups for soliton equations. In: Jimbo M, Miwa T ed. Nonlinear Integrable Systems-Classical and Quantum Theory. Singapore: World Scientific, 1983, 39-119

[17] Dickey L A. Soliton equations and Hamiltonian systems (2nd Edition). Singapore: World Scintific, 2003

[18] Date E, Kashiwara M, Jimbo M, Miwa T. KP hierarchy of orthogonal symplectic type transformation groups for soliton equations VI. J Phys Soc Japan, 1981, 50: 3813-3818

[19] Kaup D J. On the inverse scattering problem for cubic eigenvalue problems of the class $\Phi_{3x} + 6Q\Phi_x + 6R\Phi = \lambda\Phi$. Stud Appl Math, 1980, 62: 189-216

[20] Kupershmidt B A. A super Korteweg-de Vires equation: an integrble system. Phys Lett A, 1984, 102: 213-215

[21] Konopelchenko B G, Sidorenko J, Strampp W. $(1+1)$-dimensional integrable systems as symmetry constraints of $(2+1)$-dimensional systems. Phys Lett A, 1991, 157: 17-21

[22] Cheng Y, Li Y S. The constraint of the Kadomtsev-Petviashvili equation and its special solutions. Phys Lett A, 1991, 157: 22-26

[23] Loris I. On reduced CKP equations. Inverse Problems, 1999, 15: 1099-1109

[24] Olver P J. Evolution equations possessing infinitely many symmetries. J Math Phys, 1977, 18: 1212-1215

[25] Ito M. An extension of nonlinear evolution equations of the KdV(mKdV) type to higher orders. J Phys Soc Jpn, 1980, 49:771-778

[26] Aratyn H, Nissimov E, Pacheva S. Constrained KP hierarchies: additional symmetries, Darboux-Bäcklund solutions and relations to multi-matrix models. Int J Mod Phys A, 1997, 12:1265-1340

[27] Adler M. On the Bäcklund transformation for the Gel'fand-Dickey equations. Commun Math Phys, 1981, 80: 517-527